# A NOTE ON BESSEL FUNCTIONS AND CONTINUED FRACTIONS



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ABSTRACT: An identity, relating a certain infinite continued fraction to the ratio of two contiguous Bessel functions, is exploited to derive the representation of a single Bessel function in terms of a series of products of continued fractions. This representation is then made the basis of a new method for computing numerically Bessel functions of the first kind. The method is seen to be especially valuable in computing Bessel functions of order much larger than unity.

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The work reported herein was carried out in RB under Foundational Research, FR-43.

The contents of this report will be incorporated in an article concerning hypergeometric functions and continued fractions which is to be submitted for publication in the open mathematical literature.

This document is intended to serve as a progress report.

The author is grateful to Dr. A. H. Van Tuyl for several discussions, and to Susan Madigosky for programming the machine calculations.

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Captain, USN
Commander

Zaka I Slawsky
Z. I. SLAWSKY
By direction

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## ILLUSTRATIONS

## INTRODUCTION

1. The purpose of this report is twofold. First we will derive several examples of a class of not widely known identities which relate Bessel functions to certain continued fractions. Second, with the aid of one of these identities we will develop a new method for the numerical generation of Bessel functions by high speed digital computers. Throughout the report our interest will center on Bessel functions of the first kind, $J_n(z)$, where n is any integer and z is an arbitrary real number, although it will be apparent that generalizations to other cylinder functions(*) of integer order and arbitrary complex argument are possible.

2. As for the numerical computation of Bessel functions, we have in mind not the preparation of tables but the calculation of $J_n(z)$ in the midst of a lengthy sequence of calculations by a digital computer where the value of z is not known ahead of time. The greatest need for such a routine arises when z and n are both large. When $n \gg z$, the computation of $J_n(z)$ may be efficiently carried out with the series representation, and when $z \gg n$, the asymptotic expansion provides the most direct method. However there is no completely reliable standard method (to our knowledge) for computing Bessel functions in the intermediate region where $z \sim n$, especially when $n > 20$. To check the accuracy of the method developed here several short tables of Bessel functions have been made and included in this report. In no case has the 6th decimal place been discovered to be incorrect.

## BESSEL FUNCTIONS AND CONTINUED FRACTIONS

3. Since the time of Bessel himself it has been known that the ratio of two contiguous Bessel functions may be written as an infinite continued fraction. This fact follows in an elementary way from the recurrence relations (Watson, ibid., pp. 45, 153), which may be rearranged in the form:

$$\frac{J_n(z)}{J_{n-1}(z)} = \frac{\frac{z}{2}\,\frac{1}{n}}{1 - \frac{z}{2}\,\frac{J_{n+1}(z)}{nJ_n(z)}} \tag{1}$$

*A cylinder function is any function satisfying the Bessel function recurrence relations. For a full description, see Watson, <u>Theory of Bessel Functions</u> (Cambridge Univ. Press, 1958), p. 82.

The extension of this identity into explicit continued fraction form is obvious. If we change from $J_n(z)$ to $J_n(2x)$ to avoid factors of 1/2 throughout, we obtain

$$\frac{J_n(2x)}{J_{n-1}(2x)} = \cfrac{\frac{x}{n}}{1 - \cfrac{x^2/n(n+1)}{1 - \cfrac{x^2/(n+1)(n+2)}{1 - \cfrac{x^2/(n+2)(n+3)}{1 - \cdots}}}} \tag{2}$$

In the shorthand notation which we will use throughout, Eq. ( 1 ) is simply

$$\frac{J_n(2x)}{J_{n-1}(2x)} = \frac{x_n}{1-F_n} , \tag{3}$$

where $x_n = \frac{x}{n}$, and $F_n$ is the specific continued fraction defined for all $n \geq 1$ by the functional relation

$$F_n = \frac{x_n x_{n+1}}{1 - F_{n+1}} \tag{4}$$

Clearly then, it is possible to represent the ratio of any two Bessel functions, $J_{m+k}(2x)/J_m(2x)$, whose orders differ by an integer, by a product of continued fractions:

$$\frac{J_{m+k}(2x)}{J_m(2x)} = \prod_{r=1}^{k} \frac{x_{m+r}}{1 - F_{m+r}} . \tag{5}$$

Such a product representation of the ratio of two Bessel functions we will denote $P_m^{m+k}(2x)$. We also define $P_m^m \equiv 1$; and, if $n > m$, $P_n^m \equiv 1/P_m^n$.

4. With these few preliminaries taken care of, we now state the identity which we will use in the numerical computations, Paragraph 10. Several generalizations of this identity will be given in the following paragraph. The identity is:

$$\frac{1}{J_n(z)} = \sum_{m=0}^{\infty} \epsilon_m P_N^{2m}(z) \quad ; \tag{6}$$

where $\epsilon_0 = 1$, $\epsilon_m = 2$ for $m \geq 1$.

A proof of Eq. ( 6 ) is arranged very easily. It depends trivially on the fact that the P's are ratios of Bessel functions, and the well-known addition theorem (Watson, ibid., p. 23):

$$\sum_{m=0}^{\infty} \epsilon_m J_{2m}(z) = 1. \tag{7}$$

Thus we have arrived at a new, or at least not widely known, result:(*) the representation of a single Bessel function, rather than the ratio of two Bessel functions, by infinite continued fractions.

5. It should be mentioned that other analogous continued-fraction representations of Bessel functions can be easily arrived at. For example, in quantum electrodynamics the summation of certain classes of Feynman diagrams can be shown(**) to lead to the following identity:

$$\frac{1}{J_n(z)^2} = \sum_{m=0}^{\infty} \epsilon_m \left[P_N^m(z)\right]^2 \quad , \tag{8}$$

in which the square of a Bessel function is given by a series of squares of products $P_N^m(z)$. This representation depends on the well-known identity (Watson, ibid., p. 358):

$$\sum_{m=0}^{\infty} \epsilon_m \left[J_m(z)\right]^2 = 1 . \tag{9}$$

Also we have:

$$\frac{1}{J_n(z)} = \frac{2}{\sin z} \sum_{m=0}^{\infty} (-1)^m P_N^{2m+1}(z) \quad , \tag{10}$$

(*)To the author's knowledge the relations given in Eqs. (6,8,10) do not appear in the published literature on Bessel functions. In view of the simplicity of derivation, however, it seems unlikely that they have heretofore gone unnoticed.

(**)Z. Fried and J. H. Eberly (paper in preparation). It was this work which led the author's attention to the subject of the present report.

and several others slightly more complicated, which we omit here, but each of which is related in an obvious way to a known summation identity like Eq. ( 7 ) or ( 9 ).

6. Some similar results appear to be possible with the Bessel functions replaced by general confluent hypergeometric functions. However these generalizations are peripheral to our purpose here, and we will restrict the present discussion to Bessel functions of integer order.

CONVERGENCE QUESTIONS

7. A very significant property, from the standpoint of numerical computation, of the sums like Eq. ( 6 ) is their quite rapid convergence. Before going further into the matter, though, several points should be made clear. In the first place, all of the infinite series and infinite continued fractions which will be dealt with do converge (Watson, ibid., pp. 153-154, et passim). The series and the continued fractions in all cases represent analytic functions of both index and argument, with simple poles, at worst, for singularities. That is, the question of convergence is settled from the outset, and our concern is entirely for the practical question of speed of convergence. Specifically, we will be interested in the answers to these two questions: Given the continued fraction representation for $J_N(z)$, Eq. ( 6 ), how many terms of the infinite series must we keep to ensure the desired numerical accuracy; and how shall we truncate the infinite continued fractions appearing in the terms of the series? Thus we are interested in making sufficiently accurate approximations to the terms in the series as well as to the series itself. We shall answer the first question first. Assume that we want to compute $J_n(z)$, and that n and z are both large and comparable in magnitude so that neither the usual series representation nor the asymptotic expression for $J_n$ are conveniently usable. Thus we turn to the series ( 6 ). From the definition of the P's in terms of ratios of Bessel functions it is clear, though, that as far as speed of convergence goes we need consider only the problem of making M large enough so that $S_M$ is sufficiently close to unity, where $S_M$ is defined to be:

$$S_M(z) = \sum_{m=0}^{M} \epsilon_m J_{2m}(z)$$

We can conclude that convergence is likely to be quite rapid after we reach the terms $2m \sim z$ from inspection of the inequality (Watson, ibid., p. 49):

$$J_{2m}(z) \leq \frac{|z/2|^{2m}}{(2m)!}$$

8. In machine summation, in which the computer is usually limited to eight significant figures, it is of course useless to consider terms which are smaller than $10^{-8}$ times the largest term in the series, so we are interested in knowing, for a given z, how big the $J_{2m}(z)$ get. Although the maximum value of $J_{2m}(z)$ is known to occur near the point $z = 2m$, the problem of determining bounds on the size of the maximum seems to be unexplored territory. Therefore we will be satisfied with the crude, but almost certainly true(*), estimate that for any $z < 100$ there will be a term in the series $S_{\infty}$, Eq. ( 7 ), greater than 0.10. This means that we need take, in the series $S_M$, M only so high that $J_{2p}(z) < 10^{-9}$ for all $p > M$. A glance at tables of Bessel functions (**) enables us to construct a graph to serve as a guide in choosing this value of M, given a value of z. Roughly speaking, the graph shows that a sufficient value of M is given by $M = \frac{3}{4} z + 6$. The graph is reproduced below in Figure 1.

9. Now we must attend to the second question, that of approximating the infinite continued fractions which appear in every term of the series ( 6 ). Notice that by repeated use of the functional relation ( 4 ) we may easily compute all of the $F_n$ if we already know one of them. Also notice that the higher the subscript n, the more rapidly $F_n$ converges. Thus our procedure is very simple: we determine what the highest occurring subscript is, compute the corresponding F from its form as a continued fraction, and then compute all other F's from it by means of the relation ( 4 ). The computation of the highest F, say $F_p$ , is straightforward. Written out explicitly, $F_p$ is:

$$F_p = \cfrac{x_p x_{p+1}}{1 - \cfrac{x_{p+1}x_{p+2}}{1 - \cfrac{x_{p+2}x_{p+3}}{1 - \cfrac{x_{p+3}x_{p+4}}{1 - \ddots}}}}$$

The first step is to compute only the first level, $x_p x_{p+1}$. If it is already less than $10^{-8}$, then $F = 0$ is a sufficiently accurate approximation. If not, then the second level is included and the fraction computed to this approximation. If the inclusion of the second level makes a difference of less than $10^{-8}$ in the

(*)This estimate was arrived at by inspection of the Tables of Bessel Functions of the First Kind (Harvard Univ. Press, 1947-1951). Here are tabulated $J_0$ to $J_{135}$ for $0 < z < 100$.
(**)Very convenient for this purpose is a table in Jahnke & Emde, Tables of Functions, 4th ed. (Dover Publ., New York, 1945), pp. 171-179.

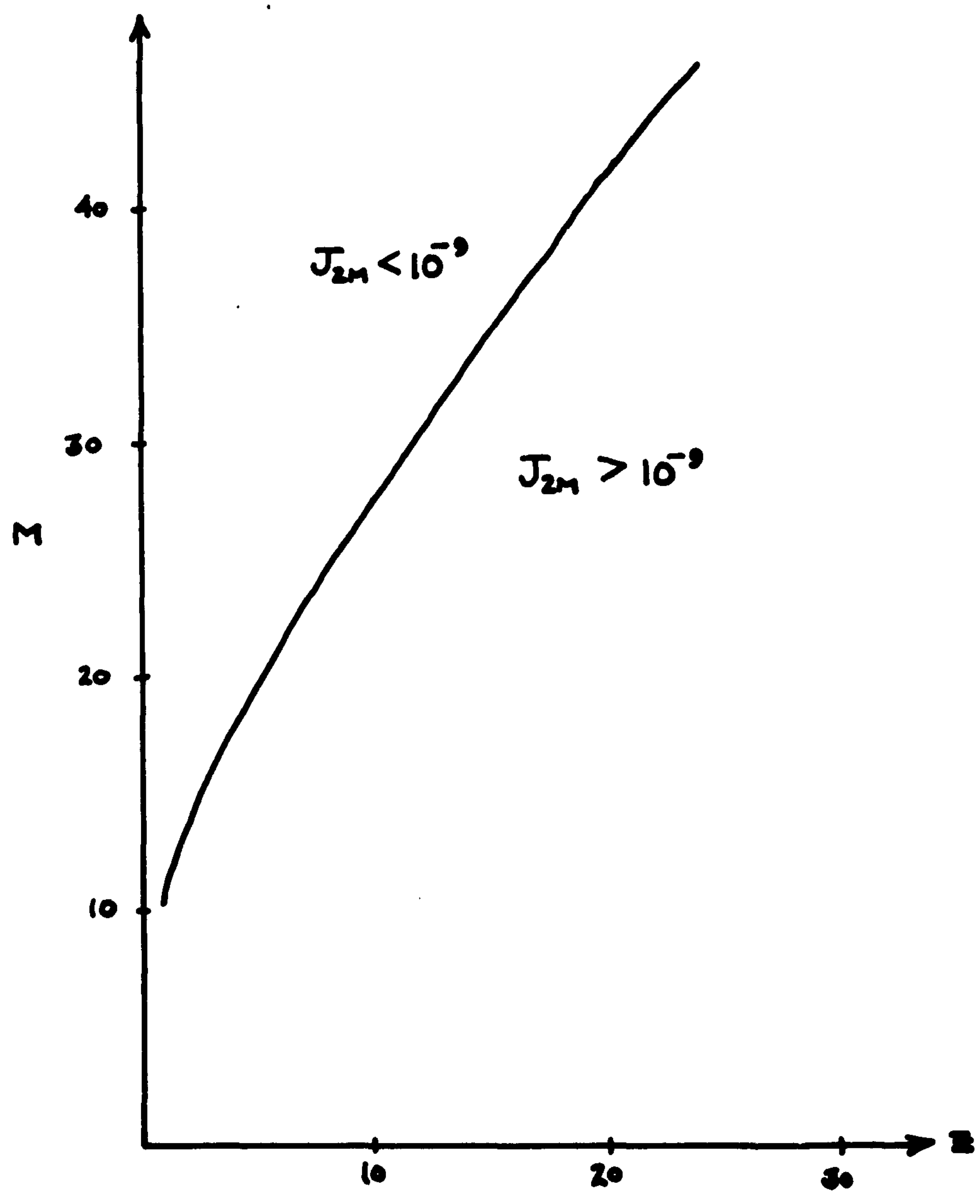


FIG. 1 VALUES OF M SUCH THAT $J_{2M}(z) < 10^{-9}$

single-level result, then $F_p = x_p x_{p+1}$ is good enough; and so on until the inclusion of one more level affects the preceding result by less than $10^{-8}$. The only remaining matter, the determination of the highest occurring subscript, is easily settled. If we are computing $J_N(z)$, and have decided to keep M terms in the series ( 6 ), then the larger of N and 2M will be the largest subscript.

## EXAMPLES OF NUMERICAL COMPUTATION

10. On the following pages we display some results obtained by machine, computation of Bessel functions using the continued fraction method described in this report. The labeling should be clear. For a given value of z several different $J_N$'s are computed; then z is changed and the $J_N$'s are computed again, and so on. Most of the values have been checked so far as possible against standard tables, in particular the Harvard tables mentioned in Paragraph 8 , and no errors have been discovered in the sixth decimal place. Only near the zeros of the $J_N(z)$ are there errors in the seventh decimal place. In at least one case, $J_{36}(10)$, there was no error until the twenty-fourth decimal place.

$l$ = 30

| | |
|---|---|
| J36 | J30 |
| 0.71294389E-02 | 0.14393584E-00 |
| J24 | J18 |
| -0.32381186E-01 | 0.15847528E-00 |
| J12 | J6 |
| 0.14925331E-00 | 0.48622734E-02 |

$l$ = 29

| | |
|---|---|
| J36 | J30 |
| 0.34371736E-02 | 0.10304804E-00 |
| J24 | J18 |
| 0.78985297E-01 | 0.14137213E-00 |
| J12 | J6 |
| 0.11998265E-00 | 0.12702166E-00 |

$l$ = 28

| | |
|---|---|
| J36 | J30 |
| 0.15476045E-02 | 0.67685383E-01 |
| J24 | J18 |
| 0.17043918E-00 | 0.39404874E-01 |
| J12 | J6 |
| -0.38292769E-02 | 0.13933615E-00 |

$l$ = 27

| | |
|---|---|
| J36 | J30 |
| 0.64974047E-03 | 0.40959226E-01 |
| J24 | J18 |
| 0.22087099E-00 | -0.89281379E-01 |
| J12 | J6 |
| -0.12946655E-00 | 0.27127728E-01 |

*l* = 26

| | |
|---|---|
| J36 | J30 |
| 0.25380371E-03 | 0.22882923E-01 |
| J24 | J18 |
| 0.22714347E-00 | -0.17521880E-00 |
| J12 | J6 |
| -0.16109037E-00 | -0.11366042E-00 |

*l* = 25

| | |
|---|---|
| J36 | J30 |
| 0.91988669E-04 | 0.11809026E-01 |
| J24 | J18 |
| 0.19977850E-00 | -0.17577267E-00 |
| J12 | J6 |
| -0.72867787E-01 | -0.15870029E-00 |

*l* = 24

| | |
|---|---|
| J36 | J30 |
| 0.30830362E-04 | 0.56256811E-02 |
| J24 | J18 |
| 0.15504220E-00 | -0.93111813E-01 |
| J12 | J6 |
| 0.72990114E-01 | -0.64547005E-01 |

*l* = 23

| | |
|---|---|
| J36 | J30 |
| 0.95162513E-05 | 0.24697721E-02 |
| J24 | J18 |
| 0.10782238E-00 | 0.34019047E-01 |
| J12 | J6 |
| 0.17301857E-00 | 0.90859228E-01 |

$l = 22$

| | |
|---|---|
| J36 | J30 |
| 0.26921725E-05 | 0.99634806E-03 |
| J24 | J18 |
| 0.67729434E-01 | 0.15492100E-00 |
| J12 | J6 |
| 0.15657871E-00 | 0.17325246E-00 |

$l = 21$

| | |
|---|---|
| J36 | J30 |
| 0.69409896E-06 | 0.36822633E-03 |
| J24 | J18 |
| 0.38570037E-01 | 0.23157261E-00 |
| J12 | J6 |
| 0.32928689E-01 | 0.10764861E-00 |

$l = 20$

| | |
|---|---|
| J36 | J30 |
| 0.16200121E-06 | 0.12401536E-03 |
| J24 | J18 |
| 0.19929105E-01 | 0.25108984E-00 |
| J12 | J6 |
| -0.11899064E-00 | -0.55086067E-01 |

$l = 19$

| | |
|---|---|
| J36 | J30 |
| 0.33960712E-07 | 0.37849146E-04 |
| J24 | J18 |
| 0.93306652E-02 | 0.22352314E-00 |
| J12 | J6 |
| -0.20545822E-00 | -0.17876715E-00 |

Z = 18

| | |
|---|---|
| J36 | J30 |
| 0.63352702E-08 | 0.10393653E-04 |
| J24 | J18 |
| 0.39458132E-02 | 0.17062988E-00 |
| J12 | J6 |
| -0.17624115E-00 | -0.15595619E-00 |

Z = 17

| | |
|---|---|
| J36 | J30 |
| 0.10401692E-08 | 0.25460067E-05 |
| J24 | J18 |
| 0.14996625E-02 | 0.11381101E-00 |
| J12 | J6 |
| -0.48574816E-01 | 0.71536081E-03 |

Z = 16

| | |
|---|---|
| J36 | J30 |
| 0.14835178E-09 | 0.55052391E-06 |
| J24 | J18 |
| 0.50874504E-03 | 0.66848081E-01 |
| J12 | J6 |
| 0.11240026E-00 | 0.16672073E-00 |

Z = 15

| | |
|---|---|
| J36 | J30 |
| 0.18091641E-10 | 0.10374711E-06 |
| J24 | J18 |
| 0.15266958E-03 | 0.34625982E-01 |
| J12 | J6 |
| 0.23666584E-00 | 0.20614970E-00 |

| | | |
|---|---|---|
| Z = 14 | J36 | J30 |
| | 0.18507218E-11 | 0.16775401E-07 |
| | J24 | J18 |
| | 0.40063892E-04 | 0.15768585E-01 |
| | J12 | J6 |
| | 0.28545026E-00 | 0.81168156E-01 |
| Z = 13 | J36 | J30 |
| | 0.15512068E-12 | 0.22828785E-08 |
| | J24 | J18 |
| | 0.90604634E-05 | 0.62693182E-02 |
| | J12 | J6 |
| | 0.26153687E-00 | -0.11803067E-00 |
| Z = 12 | J36 | J30 |
| | 0.10345578E-13 | 0.25522593E-09 |
| | J24 | J18 |
| | 0.17332263E-05 | 0.21522497E-02 |
| | J12 | J6 |
| | 0.19528018E-00 | -0.24372476E-00 |
| Z = 11 | J36 | J30 |
| | 0.52908568E-15 | 0.22735386E-10 |
| | J24 | J18 |
| | 0.27382811E-06 | 0.62803937E-03 |
| | J12 | J6 |
| | 0.12159979E-00 | -0.20158397E-00 |

Z = 10

| | |
|---|---|
| J36 | J30 |
| 0.19782072E-16 | 0.15510963E-11 |
| J24 | J18 |
| 0.34632634E-07 | 0.15244249E-03 |
| J12 | J6 |
| 0.63370255E-01 | -0.14458822E-01 |

Z = 9

| | |
|---|---|
| J36 | J30 |
| 0.50778833E-18 | 0.76921580E-13 |
| J24 | J18 |
| 0.33643764E-08 | 0.29878891E-04 |
| J12 | J6 |
| 0.27392890E-01 | 0.20431653E-00 |

Z = 8

| | |
|---|---|
| J36 | J30 |
| 0.82174473E-20 | 0.25831003E-14 |
| J24 | J18 |
| 0.23727489E-09 | 0.45380943E-05 |
| J12 | J6 |
| 0.96238223E-02 | 0.33757590E-00 |

Z = 7

| | |
|---|---|
| J36 | J30 |
| 0.74386530E-22 | 0.53172617E-16 |
| J24 | J18 |
| 0.11221933E-10 | 0.50369681E-06 |
| J12 | J6 |
| 0.26556201E-02 | 0.33919660E-00 |

Z=28.100

| | |
|---|---|
| J25 | J24 |
| 0.21628813E-00 | 0.16293546E-00 |
| J23 | J22 |
| 0.62035803E-01 | -0.61382186E-01 |
| J21 | J20 |
| -0.15815026E-00 | -0.17499898E-00 |

Z=28.200

| | |
|---|---|
| J25 | J24 |
| 0.21311807E-00 | 0.15501857E-00 |
| J23 | J22 |
| 0.50743329E-01 | -0.72245761E-01 |
| J21 | J20 |
| -0.16346721E-00 | -0.17121604E-00 |

Z=28.300

| | |
|---|---|
| J25 | J24 |
| 0.20950394E-00 | 0.14670326E-00 |
| J23 | J22 |
| 0.39321386E-01 | -0.82788636E-01 |
| J21 | J20 |
| -0.16803870E-00 | -0.16659742E-00 |

Z=28.400

| | |
|---|---|
| J25 | J24 |
| 0.20544336E-00 | 0.13800622E-00 |
| J23 | J22 |
| 0.27806593E-01 | -0.92967365E-01 |
| J21 | J20 |
| -0.17184053E-00 | -0.16116301E-00 |

Z=28.500

| J25 | J24 |
|---|---|
| 0.20093555E-00 | 0.12894613E-00 |

| J23 | J22 |
|---|---|
| 0.16236876E-01 | -0.10273923E-00 |

| J21 | J20 |
|---|---|
| -0.17485183E-00 | -0.15493716E-00 |

Z=28.600

| J25 | J24 |
|---|---|
| 0.19598127E-01 | 0.11954355E-00 |

| J23 | J22 |
|---|---|
| 0.46512783E-02 | -0.11206247E-00 |

| J21 | J20 |
|---|---|
| -0.17705508E-00 | -0.14794850E-00 |

Z=28.700

| J25 | J24 |
|---|---|
| 0.19058290E-00 | 0.10982095E-00 |

| J23 | J22 |
|---|---|
| -0.69102218E-02 | -0.12089656E-00 |

| J21 | J20 |
|---|---|
| -0.17843643E-00 | -0.14022993E-00 |

Z=28.800

| J25 | J24 |
|---|---|
| 0.18474452E-00 | 0.99802676E-01 |

| J23 | J22 |
|---|---|
| -0.18406720E-01 | -0.12920229E-00 |

| J21 | J20 |
|---|---|
| -0.17898568E-00 | -0.13181849E-00 |

Z=28.900

| | |
|---|---|
| J25 | J24 |
| C.17847197E-CC | 0.89514854[illegible]-C1 |
| J23 | J22 |
| -0.29796767E-C1 | -0.13694223E-CO |
| J21 | J20 |
| -0.1786966CE-CC | -C.12275526E-CO |

Z=29.000

| | |
|---|---|
| J25 | J24 |
| 0.17177283E-CC | C.78985407E-C1 |
| J23 | J22 |
| -C.41038357E-C1 | -0.14408C73E-CO |
| J21 | J20 |
| -0.17756689E-CC | -C.11308513[illegible]-00 |

Z=29.10C

| | |
|---|---|
| J25 | J24 |
| 0.16465648E-CC | 0.68243892E-C1 |
| J23 | J22 |
| -0.52089236E-C1 | -0.15058426E-CO |
| J21 | J20 |
| -0.17559832E-CC | -C.10285661E-00 |

Z=29.200

| | |
|---|---|
| J25 | J24 |
| C.15713416E-CC | 0.57321506E-C1 |
| J23 | J22 |
| -0.62907C39E-C1 | -0.15642163E-CO |
| J21 | J2C |
| -0.1727968CE-CO | -0.92121730E-C1 |

Z=29.300

| | |
|---|---|
| J25 | J24 |
| 0.14921893E-CC | 0.4625C942E-C1 |
| J23 | J22 |
| -0.73449466E-C1 | -0.16156410E-CO |
| J21 | J20 |
| -0.16917239E-CC | -0.80935597E-01 |

Z=29.400

| | |
|---|---|
| J25 | J24 |
| 0.14092565E-CO | 0.35066259E-01 |
| J23 | J22 |
| -0.83674617E-C1 | -C.16596573E-00 |
| J21 | J20 |
| -0.16473942E-CO | -0.69356304F· 01 |

Z=29.5C0

| | |
|---|---|
| J25 | J24 |
| 0.13227113E-CC | 0.238C2924E-C1 |
| J23 | J22 |
| -0.93540946E-C1 | -C.16966338E-00 |
| J21 | J20 |
| -0.15951632E-CO | -0.57444610E-C1 |

Z=29.600

| | |
|---|---|
| J25 | J24 |
| 0.12327393E-CC | 0.12497460E-C1 |
| J23 | J22 |
| -0.10300777E-CO | -0.17257711E-CO |
| J21 | J2C |
| -0.15352579E-CC | -0.45263553E-01 |

Z=29.700

| J25 | J24 |
|---|---|
| 0.11395445E-C0 | 0.11874785E-02 |
| J23 | J22 |
| -0.11203530E-CC | -C.17471017E-C0 |
| J21 | J20 |
| -0.14679459E-C0 | -0.32878167E-C1 |

Z=29.800

| J25 | J24 |
|---|---|
| 0.10433488E-C0 | -0.10088532E-01 |
| J23 | J22 |
| -0.12058486E-CC | -C.17604918E-C0 |
| J21 | J20 |
| -0.13935354E-C0 | -C.20355158E-C1 |

Z=29.900

| J25 | J24 |
|---|---|
| 0.94439177E-C1 | -0.21291284E-01 |
| J23 | J22 |
| -0.12861916E-CC | -0.17658436E-C0 |
| J21 | J20 |
| -0.13123744E-CC | -C.77625639E-02 |

Z=30.000

| J25 | J24 |
|---|---|
| 0.84292965E-C1 | -C.32380977E-C1 |
| J23 | J22 |
| -0.13610253E-CC | -0.17630958E-C0 |
| J21 | J20 |
| -0.12248487E-CC | C.48307385E-02 |

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| DESCRIPTORS | CODES | DESCRIPTORS | CODES | DESCRIPTORS | CODES |
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| Bessel functions | BESS | Infinite | IFIT | | |
| Continued | CONU | Ratio | RATI | | |
| Fractions | FRCT | Convergence | CNVG | | |
| Mathematics | MATH | Physics | PHYS | | |
| Method | METD | Tables | TABL | | |
| Numerical | NUMB | | | | |
| Generation | GENO | | | | |
| High speed | HIGS | | | | |
| Digital | DIGI | | | | |
| Computers | COMP | | | | |
| Computation | COMA | | | | |
| Identity | IDEN | | | | |

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